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Centre Number | Candidate Number

# Pearson Edexcel Level 3 GCE

**Time** 1 hour 30 minutes

**Paper reference** **9FM0/01**

# Further Mathematics

## Advanced

## PAPER 1: Core Pure Mathematics 1

**You must have:**
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

**Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

## Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

## Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 10 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

## Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

*Turn over*

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Q:1/1/1/1/

P71776A0132

**1.** $\mathrm{f}(z) = z^3 + az + 52$ where $a$ is a real constant

Given that $2 - 3\mathrm{i}$ is a root of the equation $\mathrm{f}(z) = 0$

(a) write down the other complex root.

**(1)**

(b) Hence

(i) solve completely $\mathrm{f}(z) = 0$

(ii) determine the value of $a$

**(4)**

(c) Show all the roots of the equation $\mathrm{f}(z) = 0$ on a single Argand diagram.

**(1)**

(a) Given one complex root of the cubic equation, $z$, we know that the second root has to be its complex conjugate, $z^*$

using rule that the complex conjugate of $z$ in the form $a+bi$ is $a-bi$,

if $z = 2-3i$,

then $z^* = 2+3i$

hence, the other complex root is $\boxed{2+3i}$

(b)(i) remembering the Fundamental Law of Algebra for cubic equation - polynomial of degree 3 has exactly 3 roots, either

· 3 real roots

· one complex conjugate pair and one real root

∴ as we've found $z$ and $z^*$, we must now find the real root of the cubic

METHOD 1: BY INSPECTION

let the real root $= \beta$

∴ can be written in factor form as $(z-\beta)$

using $z$ and $z^*$ from (a)

$$(2+3i)(2-3i)(z-\beta) = z^3 + az + 52$$

expanding the complex brackets first using

$z^2 - (\text{sum of roots})z + (\text{product of roots})$

Sum of roots: $z + z^* = 2(a_z)$ (where $z = \overset{a}{2} - \overset{b}{3i}$



P 7 1 7 7 6 A 0 2 3 2

$= 2(2)$
$= 4$

product of roots : $zz^* = a_z^2 + b_z^2$
$= -3^2 + 2^2$
$= 13$

$\therefore (z^2-4z+13)(z-\beta) \equiv z^3 + az + 52$

now expand L.H.S further

$z^3 - 4z^2 + 13z - \beta z^2 + 4z\beta - 13\beta = z^3 + az$

Collecting like terms

$z^3 - z^2(4+\beta) + z(13+4\beta) - 13\beta \equiv z^3 + az + 52$

COMPARING COEFFICIENTS

$\ldots z^2:$
$-(4+\beta) = 0$
$-4-\beta = 0$
$\Rightarrow \beta = -4$

OR

…integer:
$-13\beta = 52$
$\Rightarrow \beta = -4$

$\therefore$ real root is $-4$

$\boxed{\text{So all 3 roots are } 2 \pm 3i, -4}$

NOTE: couldn't compare 'z' coefficient as unknown 'a' in the RHS of the equation

METHOD 2: using 'roots of polynomials' formulae

Let the three roots be $\alpha, \beta$ and $\gamma$
where $\alpha = 2-3i$
$\beta = 2+3i$
$\gamma = ?$

we are given the equation of the cubic in the form

$az^3 + bz^2 + cz + d = 0$

where $a=1$, $b=0$, $c=a$ and $d=52$

remembering the 'roots of polynomials formulae

- sum of roots: $\Sigma\alpha = -b/a$
- sum of product pairs: $\Sigma\alpha\beta = c/a$
- product of pairs: $\alpha\beta\gamma = -d/a$

from the equation we can utilise the fact that b=

using associated roots of polynomial form

$\Sigma\alpha = -b/a$
$\Sigma\alpha = 0/1$
$\Sigma\alpha = 0$
$\underline{2-3i} + \underline{2+3i} + \gamma = 0$
using $z + z^* = 2a_z$
$4 + \gamma = 0$
$\Rightarrow \gamma = -4$

OR could've used fact that $d=0$
using associated roots of polynomials formula

$$\alpha\beta\gamma = -d/a$$
$$\alpha\beta\gamma = -52/1$$
$$= -52$$

$$(2-3i)(2+3i)\gamma = -52$$

using $zz^* = a_z^2 + b_z^2$

$$13\gamma = -52$$
$$\Rightarrow \gamma = -4$$

$\boxed{\text{so roots are } 2 \pm 3i, -4}$

(b)(ii) finding 'a' in a couple of ways

WAY 1: expanding QUADRATIC and LINEAR FACTORS

know from part (b)(ii) that:

$$(z^2 - 4z + 13)(z+4) = z^3 + az + 52$$

expand L.H.S

$$z^3 - \cancel{4}z^2 + 13z + \cancel{4}z^2 - 16z + 52$$

collect like terms

$$z^3 - 3z + 52 \equiv z^3 + az + 52$$

comparing 'z' coefficient

$$-3 = a$$

$$\boxed{\therefore a = -3}$$

WAY 2: USING 'ROOTS OF POLYNOMIALS' formula

from part (b)(i) we know that $c = a$,

using the associated roots of polynomials formula

$$\Sigma\alpha\beta = c/1$$
$$\Sigma\alpha\beta = a$$

$$(2+3i)(2-3i) + (-4)(2+3i) + (-4)(2-3i) = a$$
$$= 13 + (-8 - \cancel{12i}) + (-8 + \cancel{12i}) = a$$
$$\Rightarrow 13 - 16 = a$$
$$\Rightarrow \boxed{a = -3}$$

WAY 3: USING FACTOR THEOREM

factor theorem states that if $(z-a)$ is a factor of $f(z)$, then $f(a)=0$

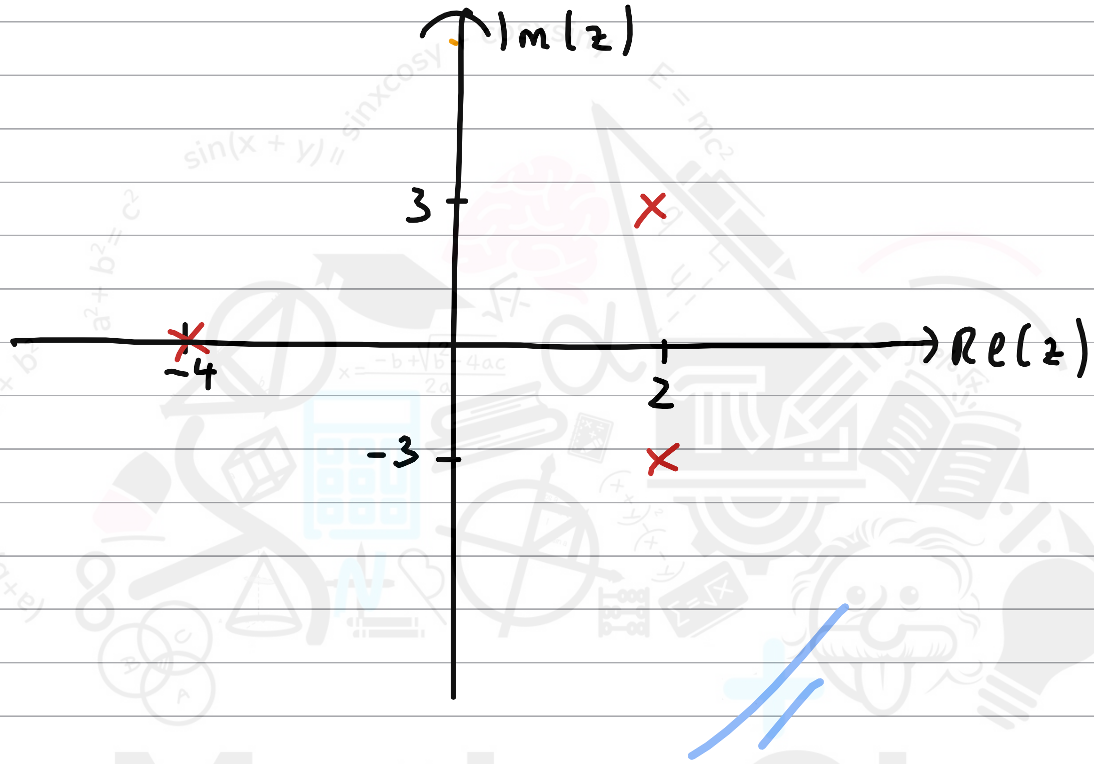
$\therefore$ using $(x-\alpha)$

$$f(-4) = (-4)^3 + a(-4) + 52 = 0$$
$$\Rightarrow -64 - 4a + 52 = 0$$
$$\Rightarrow -12 - 4a = 0$$
$$\Rightarrow 4a = -12$$
$$\Rightarrow \boxed{a = -3}$$

**Question 1 continued**

(c) need to represent $2+3i$, $2-3i$, $-4$ on an Argand diagram
remember how complex numbers $a \pm bi$ are represented as $(a, b)$ on an Argand diagram

$\therefore$ $2-3i$ represented as $(2,-3)$
$2+3i$ represented as $(2,3)$
$-4$ represented as $(-4,0)$






**(Total for Question 1 is 6 marks)**

**2.** **In this question you must show all stages of your working.**

**Solutions relying entirely on calculator technology are not acceptable.**

Determine the values of $x$ for which

$$64\cosh^4 x - 64\cosh^2 x - 9 = 0$$

Give your answers in the form $q\ln 2$ where $q$ is rational and in simplest form.

**(4)**

METHOD 1 : 'QUADRATIC IN DISGUISE'

recognise this as a quadratic in $\cosh^2 x$, hence using a $y$ substitution to solve for $x$

let $y = \cosh^2 x$

$\therefore \; 64y^2 - 64y - 9 = 0$

using equation solver on calc OR quadratic formula

$$y = \frac{64 \pm \sqrt{(64)^2 - 4(-9\times 64)}}{128} = \frac{64 \pm \sqrt{6400}}{128}$$

$$= \frac{64 \pm 80}{128}$$

$$\Rightarrow y = \frac{144}{128} = \frac{9}{8}$$

$$\text{OR } y = \frac{-16}{128} = -1/8$$

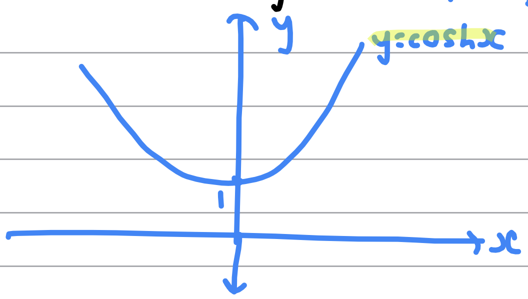
subbing $\cosh^2 x$ back in

$\cosh^2 x = 9/8$ or $-1/8$

↳ note cannot square root a negative, $\therefore$ reject this solution

$$\cosh x = \pm\sqrt{9/8}$$

but remembering the shape of the $y = \cosh x$ function




$\Rightarrow \cosh x \geq 1$

$\therefore$ rejecting -ve solution

get $\cosh x = 9/8$

now we have 2 ways to find $x$

WAY 1: using inverse function

using inverse cosh function formula

**Hyperbolic functions**

$\cosh^2 x - \sinh^2 x = 1$

$\sinh 2x = 2\sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\operatorname{arcosh} x = \ln\{x + \sqrt{x^2 - 1}\} \quad (x \geqslant 1)$

$\operatorname{arsinh} x = \ln\{x + \sqrt{x^2 + 1}\}$

$\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1)$

$x = \operatorname{arcosh}(\sqrt{9/8})$

$= \pm \ln\left(\sqrt{9/8} + \sqrt{(\sqrt{9/8})^2 - 1}\right)$

$= \pm \ln\left(\sqrt{9/8} + \sqrt{9/8 - 1}\right)$

$= \pm \ln\left(\sqrt{9/8} + \sqrt{1/8}\right)$

$= \pm \ln\left(4/\sqrt{8}\right)$

using the rationalised form of the fraction

$= \pm \ln\left(\frac{4\sqrt{2}}{4}\right)$

$= \pm \ln(\sqrt{2})$

but the question asks for $x$ in form $a\ln 2$, so rewriting $\pm\ln(\sqrt{2})$ as $\pm\ln 2^{1/2}$

and using power log law, i.e $\log m^n = n\log m$

$\boxed{x = \pm 1/2 \ln 2}$

WAY 2: using exponential definition of $\cosh x$

we know that $\cosh x = \frac{1}{2}(e^x + e^{-x})$

subbing this into $\cosh x = \sqrt{8/9} = \frac{3\sqrt{2}}{4}$

$\frac{1}{2}(e^x + e^{-x}) = \frac{3\sqrt{2}}{4}$

$\times 2 \qquad \times 2$

$e^x + e^{-x} = \frac{3\sqrt{2}}{2}$

$e^x + \frac{1}{e^x} = \frac{3\sqrt{2}}{2}$

$\times e^x \qquad \times e^x$

$e^{2x} + 1 = \frac{3\sqrt{2}}{2}e^x$

taking to one side

$e^{2x} - \frac{3\sqrt{2}}{2}e^x + 1 = 0$

notice that this is a 'quadratic in disguise' (quadratic in $e^x$) so solving using a 'y' substitution

let $y = e^x$

$y^2 - \frac{3\sqrt{2}}{2}y + 1 = 0$

using equation solver on calculator

$y = \sqrt{2}$ or $\sqrt{2}/2$

sub $e^x$ back in

$e^x = \sqrt{2}$ or $1/\sqrt{2}$

for $x$, taking natural logs of both sides

$\Rightarrow x = \ln\sqrt{2}$

$\Rightarrow x = \ln 2^{1/2}$

$\therefore x = 1/2 \ln 2$

OR

$x = \ln 1/\sqrt{2}$

$= \ln(2)^{-1/2}$

$\therefore x = -\frac{1}{2}\ln(2)$

$\boxed{\therefore x = \pm\frac{1}{2}\ln(2)}$

## METHOD 2: using HYPERBOLIC IDENTITIES

noticing that the $64\cosh^2 x$ is common for the first two terms
$\therefore$ factorising it out

$$64\cosh^2 x(\cosh^2 x - 1) - 9 = 0$$

now using the main hyperbolic identity

$$\cosh^2 x - \sinh^2 x = 1$$ (derived from parametrised equation for a hyperbola)

and rearranging to give

$$\cosh^2 x - 1 = \sinh^2 x$$

now we can replace bracket with $\sinh^2 x$

$$64\cosh^2 x(\sinh^2 x) - 9 = 0$$

$$\Rightarrow 64\cosh^2 x\sinh^2 x - 9 = 0$$

using the hyperbolic version for the sin double angle formula

$$\sinh 2x = 2\sinh x\cosh x$$

$$\sinh^2 2x = 4\sinh^2 x\cosh^2 x$$

$$16(4\sinh^2 x\cosh^2 x) = 9$$

$$16(\sinh^2 2x) = 9$$

$$\Rightarrow \sinh^2 2x = 9/16$$

square rooting both sides

$$\sinh 2x = \pm\sqrt{9/16} = \pm 3/4$$

using inverse sinh formula in formula book

$$\operatorname{arsinh}(x) = \ln(x + \sqrt{x^2+1})$$

$$2x = \operatorname{arsinh}(3/4)$$

subbing $x = 3/4$ into above formula

$$2x = \ln\left(3/4 + \sqrt{(3/4)^2 + 1}\right)$$
$$= \ln\left(3/4 + \sqrt{9/16 + 1}\right)$$
$$= \ln\left(3/4 + \sqrt{25/16}\right)$$
$$= \ln(3/4 + 5/4)$$
$$= \ln(8/4)$$
$$\div 2 \quad = \ln 2$$
$$\therefore x = \frac{1}{2}\ln 2$$

now subbing in $x = -3/4$

$$2x = \ln\left(-3/4 + \sqrt{(-3/4)^2 + 1}\right)$$
$$= \ln\left(-3/4 + \sqrt{25/16}\right)$$
$$= \ln(-3/4 + 5/4)$$
$$= \ln(2/4)$$
$$= \ln(1/2)$$
$$\Rightarrow 2x = \ln(2^{-1})$$

$$2x = -\ln(2)$$
$$\div 2 \quad x = -\tfrac{1}{2}\ln(2) \quad \div 2$$
$$\boxed{\therefore x = \pm\tfrac{1}{2}\ln(2)}$$

## METHOD 3: using exponential definition of cosh(x)

rewriting given equation using the exponential dfn of $\cosh x$

$$\cosh x = \tfrac{1}{2}(e^{x}+e^{-x})$$

$$64\left(\tfrac{1}{2}(e^{x}+e^{-x})\right)^4 - 64\left(\tfrac{1}{2}(e^{x}+e^{-x})\right)^4 - 9 = 0$$

expanding through

$$\frac{64}{2^4}(e^{x}+e^{-x})^4 - \frac{64}{2^4}(e^{x}+e^{-x})^2 - 9 = 0$$

using binomial expansion for the exponential terms

$$\frac{64}{16}(e^{4x} + 4e^{3x}e^{-x} + 6e^{2x}e^{-2x} + 4e^{x}e^{-3x} + e^{-4x}) - \frac{64}{16}(e^{2x} + 2e^{x}e^{-x} + e^{-2x}) - 9 = 0$$

expand and simplify

$$4e^{4x} + \cancel{16e^{2x}} + 24 + \cancel{16e^{-2x}} + 4e^{-4x} - \cancel{16e^{2x}} - 32 - \cancel{16e^{-2x}} - 9 = 0$$

collecting like terms

$$4e^{4x} - 17 + 4e^{-4x} = 0$$

$$4e^{4x} - 17 + \frac{4}{e^{4x}} = 0$$

$\times e^{4x}$ $\quad\quad$ $\times e^{4x}$

$$4e^{8x} - 17e^{4x} + 4 = 0$$

realising this is a 'quadratic in disguise' (in $e^{4x}$)

$\therefore$ using a y substitution to get $x$

let $y = e^{4x}$

$$4y^2 - 17y + 4 = 0$$

using equation solver calc or quadratic formula

$$y = \frac{17 \pm \sqrt{(-17)^2 - 4(4)(4)}}{8}$$

$$= \frac{17 \pm \sqrt{289 - 64}}{8}$$

$$= \frac{17 \pm 15}{8}$$

$$\therefore y = \frac{32}{8} = 4 \text{ or } \tfrac{1}{4}$$

subbing $e^{4x}$ back in

$$e^{4x} = 4$$

taking logs of both sides

$$4x = \ln 4 = \ln 2^2$$
$$4x = 2\ln 2$$
$$x = \tfrac{1}{2}\ln 2$$

**Question 2 continued**

now $e^{4x} = 1/4$

taking logs of both sides

$4x = \ln(1/4)$

$4x = \ln(4^{-1})$

$4x = -2\ln(2)$

$\div 4 \qquad \div 4$

$\Rightarrow x = -\frac{1}{2}\ln 2$

$\boxed{\therefore x = \pm\frac{1}{2}\ln 2}$



**(Total for Question 2 is 4 marks)**

**3.** (a) Determine the general solution of the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = e^{2x} \cos^2 x$$

giving your answer in the form $y = f(x)$

**(3)**

Given that $y = 3$ when $x = 0$

(b) determine the smallest positive value of $x$ for which $y = 0$

**(3)**

(a) · straight away we can see that we cannot solve this first order differential equation by separation of variables as it involves an addition rather than a product of 2 variables and $\frac{dy}{dx}$

· next, see if we can use the reverse ~~chain~~ product rule

... considering the LHS of the equation ·

$\frac{d}{dx}(y) = \frac{dy}{dx}$ ✓

$$\cos x \frac{dy}{dx} + y \sin x$$

$\frac{d}{dx}(\cos x) \neq \sin x$ ✗

∴ not reverse product rule

· hence, the only way is through introducing an integration factor - but first, need to get equation in the form

$$\frac{dy}{dx} + Py = Q$$

∴ need to divide through by $\cos x$

$$\cos x \frac{dy}{dx} + y \sin x = e^{2x} \cos^2 x$$

$\div \cos x$ $\qquad \div \cos x$

$$\frac{dy}{dx} + y \tan x = e^{2x} \cos x$$

now introducing the I.F $= e^{\int P dx}$

$$e^{\int \tan x \, dx} = e^{\ln|\sec x| dx} = \sec x$$

multiplying through by $\sec x$

$\times \sec x$ $\qquad \frac{d}{dx}(y) = \frac{dy}{dx}$ ✓ $\qquad \times \sec x$

$$\frac{dy}{dx} \sec x + y \tan x \sec x = e^{2x}$$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ ✓

now we can check for reverse product rule (above) ✓

∴ can rewrite equation as the derivative of the product of $\sec x$ and $y$

$$\frac{d}{dx}(y\sec x) = e^{2x}$$

integrating both sides and solving for y

$$\int \frac{d}{dx}(y\sec x)\,dx = \int e^{2x}\,dx$$

$$y\sec x = e^{2x} + c$$

$\div \sec x$ $\div \sec x$

$$y = \frac{e^{2x}}{2\sec x} + \frac{c}{\sec x}$$

OR

$$y = \frac{1}{2}\cos x\, e^{2x} + \cos x\, c$$

$$\boxed{y = \cos x\left(\tfrac{1}{2}e^{2x} + c\right)}$$

(b) ∴ general solution to the differential equation is

$$y = \cos x\left(\tfrac{1}{2}e^{2x} + c\right)$$

where different values for 'c' give different particular solutions

here we're given the values needed to get the 'c' for one particular solution

subbing in $x=0, y=3$ into general equation

$$3 = \cos(0)\left(\tfrac{1}{2}e^{2(0)} + c\right)$$

$$\Rightarrow 3 = 1\left(\tfrac{1}{2}(1) + c\right)$$

$$\Rightarrow 3 = \tfrac{1}{2} + c$$

$$\Rightarrow c = \tfrac{5}{2}$$

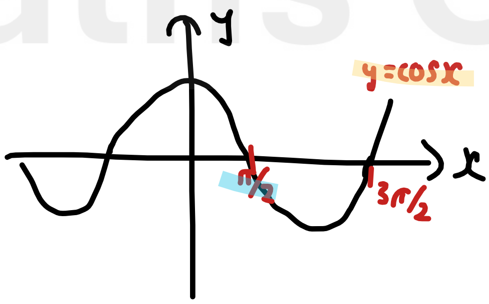
∴ particular solution is

$$y = \cos x\left(\tfrac{1}{2}e^{2x} + \tfrac{5}{2}\right)$$

now considering where $y=0$

$$0 = \cos x\left(\tfrac{1}{2}e^{2x} + \tfrac{5}{2}\right)$$

∴ where $\cos x = 0$




$$\boxed{\therefore x = \pi/2}$$

(Total for Question 3 is 6 marks)

**4.** (a) Use the method of differences to prove that for $n > 2$

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) \equiv \ln\left(\frac{n(n+1)}{2}\right)$$

**(4)**

(b) Hence find the exact value of

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$$

Give your answer in the form $a\ln\left(\frac{b}{c}\right)$ where $a$, $b$ and $c$ are integers to be determined.

**(3)**

(a) remembering the general formula needed for finding summations using method of differences

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} f(r) - f(r+1)$$

in this question, $r=2$ and $u_r = \ln\left(\frac{r+1}{r-1}\right)$

∴ need to change 'r' into $r=2$ and use the quotient log law to get a 'difference' of two functions of 'r'

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) = \sum_{r=2}^{n} \ln(r+1) - \ln(r-1)$$

couple of ways to evaluate the summation

WAY 1 : NUMERICAL METHOD

evaluate above for $r = 2, 3, 4, (\ldots), n-2, n-1$ and '$n$'

$u_2 : \cancel{\ln(3)} \boxed{-\ln(1)}$
$+$
$u_3 : \cancel{\ln(4)} \boxed{-\ln(2)}$
$+$
$u_4 : \cancel{\ln(5)} - \cancel{\ln(3)}$

$\vdots$

$u_{n-2} : + \cancel{\ln(n-1)} - \cancel{\ln(n-3)}$
$u_{n-1} : + \boxed{\ln(n)} - \cancel{\ln(n-2)}$
$u_n : + \boxed{\ln(n+1)} - \cancel{\ln(n-1)}$

notice the CANCELLING terms



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left with non-cancelled terms

undefined

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) = \ln(n)+\ln(n+1)-\ln(1)-\ln(2)$$

$$= \ln(n)+\ln(n+1)-\ln(2)$$

using product and quotient log laws

$$\boxed{= \ln\left(\frac{n(n+1)}{2}\right)}$$

as required

WAY 2: MECHANICAL METHOD

recognise

$$\sum_{r=2}^{n} \ln(r+1)-\ln(r-1)$$

more generally as:

$$\sum_{r=2}^{n} f(r+2)-f(r)$$

where $f(r)=\ln(r-1)$

and $f(r+2)=\ln(r+1)$

evaluating above for $r=2,3,4\,(\ldots),n-2,n-1,n$

$u_2:\ \cancel{f(4)}-f(2)$

$u_3:+\ \cancel{f(5)}-f(3)$

$u_4:+\ f(6)-\cancel{f(4)}$

$\vdots$

$u_{n-2}:+\ \cancel{f(n)}-\cancel{f(n-2)}$

$u_{n-1}:+\ f(n+1)-\cancel{f(n-1)}$

$u_n:+\ f(n+2)-\cancel{f(n)}$

notice the CANCELLING functions

left with non-cancelled functions

$$\sum_{r=2}^{n} \ln(r+1)-\ln(r-1) = f(n+1)+f(n+2)-f(2)-f(3)$$

subbing into above definitions

undefined

$$\ln(n+1)+\ln(n)-\ln(2)-\ln(1)$$

using product and quotient log laws

$$\boxed{\Rightarrow \ln\left(\frac{n(n+1)}{2}\right)}$$

**Question 4 continued**

(b) notice the difference between part (a) and part (b): $n=100$, $n=51$ and whole summation is raised to a power of 35

let's deal with the power of 35 first

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)^{35}$$

using power log law, can bring the power as a coefficient in front of the 'sigma' notation

i.e $$35\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right)$$

Now need $r=2$, using the formula for when $r\neq 1$

$$\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$$

but with $r=2$

$$35\left(\sum_{r=2}^{100} \ln\left(\frac{r+1}{r-1}\right) - \sum_{r=2}^{50} \ln\left(\frac{r+1}{r-1}\right)\right)$$

and subbing in $n=100$ into first and $n=50$ into second

$$35\left(\ln\left(\frac{100\times 101}{2}\right) - \ln\left(\frac{50\times 51}{2}\right)\right)$$

$$= 35\left(\ln(5050) - \ln(1275)\right)$$

using power log law

$$= 35\left(\ln\left(\frac{5050}{1275}\right)\right)$$

$$\boxed{= 35\ln\left(\frac{202}{51}\right)}$$

(Total for Question 4 is 7 marks)

**5.**

$$\mathbf{M} = \begin{pmatrix} a & 2 & -3 \\ 2 & 3 & 0 \\ 4 & a & 2 \end{pmatrix} \quad \text{where } a \text{ is a constant}$$

(a) Show that **M** is non-singular for all values of $a$.

**(2)**

(b) Determine, in terms of $a$, $\mathbf{M}^{-1}$

**(4)**

(a) We know that, to show that the matrix M is non-singular, we need to show that $\det(M) \neq 0$ (i.e if DOES have an inverse)

using the formula for det. of 3×3 matrices

where

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

subbing in values for matrix M

$$\det(M) = a\begin{vmatrix} 3 & 0 \\ a & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 4 & a \end{vmatrix}$$

$$= a(6-0) - 2(4-0) - 3(2a-12)$$

$$= \cancel{6a} - 8 - \cancel{6a} + 36$$

$$= 28 \neq 0$$

$\therefore$ non-singular

(b) usually can calculate $M^{-1}$ using a calculator - but here the unknown 'a' requires us to find the inverse manually

Step 1 : $\dfrac{1}{\det(M)} = \dfrac{1}{28}$

Step 2 : matrix of MINORS (the det. of the 2×2 matrix left after deleting all rows and columns that contain the element)

**Question 5 continued**

$$M = \begin{pmatrix} 6-0 & 4-0 & 2a-12 \\ 4-(-3a) & 2a+12 & a^2-8 \\ 0+9 & 0+6 & 3a-4 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 2a-12 \\ 4+3a & 2a+12 & a^2-8 \\ 9 & 6 & 3a-4 \end{pmatrix}$$

step 3: form a matrix of cofactors (change the sign of those with -ve)

$$\begin{pmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{pmatrix} \qquad \therefore C = \begin{pmatrix} 6 & -4 & 2a-12 \\ -4-3a & 2a+12 & -a^2+8 \\ 9 & -6 & 3a-4 \end{pmatrix}$$

step 4: transpose the matrix of cofactors by switching places of the following:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \therefore C^T = \begin{pmatrix} 6 & -4-3a & 9 \\ -4 & 2a+12 & -6 \\ 2a-12 & 8-a^2 & 3a-4 \end{pmatrix}$$

Step 5: $M^{-1} = \frac{1}{\det(A)} C^T$

$$\boxed{= \frac{1}{28}\begin{pmatrix} 6 & -4-3a & 9 \\ -4 & 2a+12 & -6 \\ 2a-12 & 8-a^2 & 3a-4 \end{pmatrix}}$$



**(Total for Question 5 is 6 marks)**

**6.** (a) Express as partial fractions

$$\frac{2x^2+3x+6}{(x+1)(x^2+4)}$$

**(3)**

(b) Hence, show that

$$\int_0^2 \frac{2x^2+3x+6}{(x+1)(x^2+4)}\,dx = \ln\left(a\sqrt{2}\right)+b\pi$$

where $a$ and $b$ are constants to be determined.

**(4)**

(a) Usually in Pure Year 2, we'd have the partial fractions as

$$\frac{2x^2+3x+6}{(x+1)(x^2+4)} = \frac{A}{x+1}+\frac{B}{x^2+4}$$

but here in FM, we need the numerators of the partial fractions to be as general as possible, so if there is a quadratic in the denominator, there has to be a linear term in the numerator

$$\frac{2x^2+3x+6}{(x+1)(x^2+4)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+4}$$

preparing to solve for 'A', 'B' and 'C'

$$2x^2+3x+6 = A(x^2+1)+(Bx+C)(x+1)$$

WAY 1: by substitution

making each bracket equal to 0

$$2x^2+3x+6 = A(x^2+4)+(Bx+C)(x+1)$$

sub in $x=-1$

$$2(-1)^2+3(-1)+6 = A((-1)^2+4)+(B(-1)+C)(-1+1)$$

$$5 = 5A$$

$$\Rightarrow A=1$$

sub in $x=0$

$$2(0)^2+3(0)+6 = A((0)^2+4)+(B(0)+C)(0+1)$$

$$6 = 4A+C$$

$$6 = 4(1)+C$$

$$\Rightarrow C=2$$

**Question 6 continued**

sub in $x = 1$

$$2(1)^2 + 3(1) + 6 = A((1)^2 + 4) + B((1) + C)(1+1)$$

$$11 = 5A + 2B + 4$$

$$11 = 5(1) + 2B + 4$$

$$\Rightarrow 2B = 2$$

$$B = 1$$

$$\boxed{\therefore \frac{2x^2+3x+6}{(x+1)(x^2+4)} = \frac{1}{x+1} + \frac{x+2}{x^2+4}}$$

WAY 2: comparing coefficients

getting the bracket form of equation

$$2x^2 + 3x + 6 = A(x^2+4) + (Bx+C)(x+1)$$

expanding RHS and collecting like terms

$$2x^2 + 3x + 6 = x^2(A+B) + x(B+C) + 4A + C$$

...comparing $x^2$: $2 = A + B$ — (1)

...comparing $x$: $3 = B + C$ — (2)

...comparing constants: $6 = 4A + C$ — (3)

using (1) and (2) to get expression for B

$$B = 2 - A$$

$$B = 3 - C$$

equating: $2 - A = 3 - C$

$$\Rightarrow C - A = 1 \quad \text{— (4)}$$

now solving (3) and (4) simultaneously

by substitution: from (4),

sub into (3): $6 = 4A + (A+1)$

$$\Rightarrow 5A = 5$$

$$\Rightarrow A = 1$$

$\therefore$ sub into (4) for $C$

$$C = 1 + 1 = 2$$

and sub into (1) for B

$$2 + 1 = B$$

$$B = -2$$

$$\boxed{\therefore \frac{2x^2+3x+6}{(x+1)(x^2+4)} = \frac{1}{x+1} + \frac{x+2}{x^2+4}}$$

(b) starting with indfnt integration of fractions using above partial fractions

$$\int \frac{2x^2+3x+6}{(x+1)(x^2+4)}\,dx = \int \frac{1}{x+1}\,dx + \int \frac{x+2}{x^2+4}\,dx$$

using $\int \frac{1}{x}\,dx = \ln|x| + c \Rightarrow \int \frac{1}{x+1}\,dx = \ln|x+1| + c$

**Question 6 continued**

now for the fractional expression, out of the three methods shown:

explained further pg.21 - end of question

**Fractional expressions**

**4a.** Can I split the numerator?
*Is there a single term in the denominator?*
**4b.** Can I do partial fractions?
*Does the denominator factorise?*
**4c.** Can I do algebraic division?
*Is the fraction improper?*

we consider 'splitting the numerator' and integrating each part separately

$$\therefore \int \frac{x+2}{x^2+4}\,dx = \boxed{\int \frac{x}{x^2+4}}\,dx + \boxed{\int \frac{2}{x^2+4}}\,dx$$

considering the integral highlighted in blue, we see that it can be integrated using reverse chain rule

$$\int \frac{x}{x^2+4}\,dx$$

consider: $\ln|x^2+4|$.

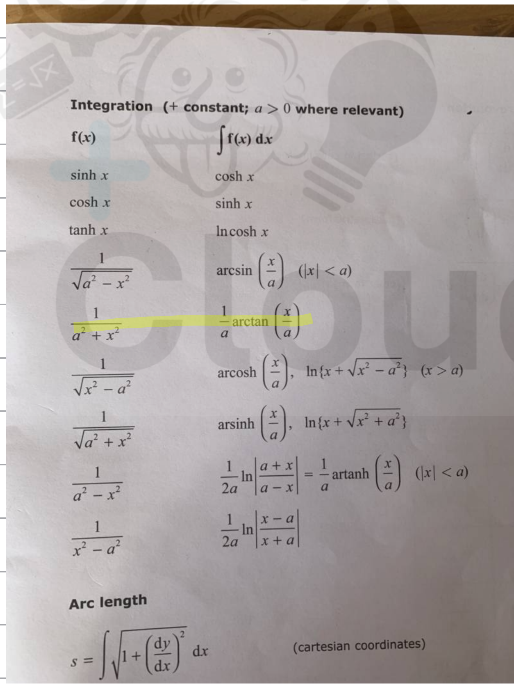
differentiate: (chain rule) $\frac{2x}{x^2+4}$

now need to scale the diff. to get integral $\times \frac{1}{2}$

$$\therefore \int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln|x^2+4| + c$$

now considering the integral highlighted pink - see we can't use any of the three methods above, so looking in our FORMULA BOOKLET for something of form $\int \frac{1}{a^2+x^2}$

use INVERSE TAN formula


Integration (+ constant; $a > 0$ where relevant)

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\ln\cosh x$ |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\arcsin\left(\frac{x}{a}\right)$ $(\lvert x\rvert < a)$ |
| $\frac{1}{a^2+x^2}$ | $\frac{1}{a}\arctan\left(\frac{x}{a}\right)$ |
| $\frac{1}{\sqrt{x^2-a^2}}$ | $\text{arcosh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2-a^2}\}$ $(x > a)$ |
| $\frac{1}{\sqrt{a^2+x^2}}$ | $\text{arsinh}\left(\frac{x}{a}\right)$, $\ln\{x+\sqrt{x^2+a^2}\}$ |
| $\frac{1}{a^2-x^2}$ | $\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\text{artanh}\left(\frac{x}{a}\right)$ $(\lvert x\rvert < a)$ |
| $\frac{1}{x^2-a^2}$ | $\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$ |

Arc length

$s = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ (cartesian coordinates)

getting our integral in a similar form

$$2\int \frac{1}{4+x^2}\,dx \qquad \left(\begin{matrix} a^2 = 4 \\ a = 2 \end{matrix}\right)$$

subbing into formula

$$2\left(\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right)$$

$$\therefore = \arctan\left(\frac{x}{2}\right) + c$$

$$\therefore \int \frac{x+2}{x^2+2} = \frac{1}{2}\ln|x^2+4| + \arctan\left(\frac{x}{2}\right) + c$$

$$\Rightarrow \int \frac{2x^2+3x+6}{(x+1)(x^2+4)} = \ln|x+1| + \frac{1}{2}\ln|x^2+4| + \arctan\left(\frac{x}{2}\right) + c$$

**Question 6 continued**

now putting the limits in

$$\left[\ln|x+1| + \frac{1}{2}\ln|x^2+4| + \arctan\left(\frac{x}{2}\right)\right]_0^2$$

$$\left\{\left[\ln|3| + \frac{1}{2}\ln|8| + \arctan(1)\right] - \left[\ln(1) + \frac{1}{2}\ln(4) + \arctan(0)\right]\right\}$$

$$= \ln|3| + \ln(2\sqrt{2}) + \pi/4 - \ln 2$$

using product then quotient log law

$$\ln\left|\frac{3\times\cancel{2}\sqrt{2}}{\cancel{2}}\right| + \pi/4$$

$$\boxed{= \ln|3\sqrt{2}| + \pi/4}$$

Reminders:

Students find fractions tough as fractions can be so many types.

Check first (and throughout the question) if you can simplify by:

- using basic indices rules to simplify and expand brackets
  - $x^a \times x^b = x^{a+b}$
  - $\frac{x^a}{x^b} = x^{a-b}$
  - $\frac{3}{5x}$ means $\frac{3}{5}x^{-1}$.
  - $(\sqrt[b]{x})^a$ or $\sqrt[b]{x^a} = x^{\frac{a}{b}}$
- Factorising and maybe cancel first
- Is there a single term in denominator?

  split fractions using $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ or $(a+b)c^{-1}$

Then ask yourself:

1. **Is it an easy power type?** $\int x^n dx = \frac{x^{n+1}}{n+1}$
2. **Is it $ln$ (natural logarithm)?** Form $\int \frac{f'(x)}{f(x)} dx$

   To recognize these, the power in the denominator is (almost always) 1. When you bring the denominator up to the numerator using negative power indices rule you get a power of$-1$. By adding one to the power and dividing it, you'll end up dividing by zero which you can't do

   $$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$$

   Method: copy ln(denominator). Remember ignore then differentiate to check you get what is inside the integral - correct with numbers only, not variables and only correct by multiplying or dividing. We can ignore the pink part since the derivative 'pops' out when we differentiate and we know when we differentiate our answer it must be what is inside the integral).
3. **Is it bring up and harder power type?** Bring the power up and becomes the form $\int f'(x) f(x)^n \, dx = \frac{f(x)^{n+1}}{n+1} + C$

   Recognisable by a power in the denominator other than

   $$\int \frac{4x}{(2x^2-1)^3} = \int 4x(2x^2-10)^{-3} \, dx \text{ etc}$$
4. **Is it Partial fractions!** Recognisable by products in the denominator.

   Form 1 $\frac{\dots}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$

   Form 2 $\frac{\dots}{(dx+e)(fx+g)^2} = \frac{A}{dx+e} + \frac{B}{fx+g} + \frac{C}{(fx+g)^2}$

   (only advanced courses have this form)

   Form 3 $\frac{\dots}{(dx+e)(fx^2+g)} = \frac{A}{dx+e} + \frac{Bx+C}{fx^2+g}$
5. **Is it divide first?** Recognisable by **two or more terms** in the denominator and also where we have the matching highest powers in both numerator and denominator or a higher power in the numerator
6. **Rewriting/adapting fraction in a clever way (split up the numerator to get two fractions)**
7. **Is it inverse trig? (may need to complete the square first)** Either use the inverse trig results below or use a trig substitution

$$\int \frac{1}{\sqrt{a^2-(bx)^2}} dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int -\frac{1}{\sqrt{a^2-(bx)^2}} dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int \frac{1}{a^2+(bx)^2} dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$$

r Question 6 is 7 marks)

**7.** Given that $z = a + bi$ is a complex number where $a$ and $b$ are real constants,

(a) show that $zz^*$ is a real number.

**(2)**

Given that

- $zz^* = 18$
- $\dfrac{z}{z^*} = \dfrac{7}{9} + \dfrac{4\sqrt{2}}{9}\mathrm{i}$

(b) determine the possible complex numbers $z$

**(5)**

(a) remembering the general expression for the complex conjugate $z^*$ of a complex number, $z$

if $z = a+bi$,

$z^* = a-bi$

=) asking for the product of these

$zz^* = (a+bi)(a-bi)$

$= a^2 + \cancel{abi} - \cancel{abi} - b^2i^2$

$= a^2 - b^2i^2$

know $i = \sqrt{-1}$

$i^2 = -1$

$\therefore a^2 - b^2(-1)$

$= a^2 + b^2$ which $\in \mathbb{R}$ (belongs to real numbers)

(b) METHOD 1: using the COMPLEX NUMBER form

starting with the fact that $zz^* = 18$

and thus subbing in $a^2+b^2$ from (a)

to get an equation with 'a' and 'b'

$a^2 + b^2 = 18$ — ①

now using $\dfrac{z}{z^*} = 7/9 + \dfrac{4\sqrt{2}}{9}i$

evaluating LHS

$\dfrac{z}{z^*} = \dfrac{a+bi}{a-bi}$ $\times a+bi$ (rationalising denominator) $\times a+bi$

$= \dfrac{a^2+2abi-b^2}{a^2+b^2} = \dfrac{7}{9} + \dfrac{4\sqrt{2}}{9}i$ — ②

**Question 7 continued**

equating real then imaginary components of equation ②

$Re\left(\frac{z}{z^*}\right) = \frac{a^2-b^2}{a^2+b^2} = 7/9$

$= \frac{a^2-b^2}{18} = 7/9$

$= a^2-b^2 = 14$ -③

$Im\left(\frac{z}{z^*}\right) = \frac{2ab}{a^2+b^2} = \frac{4\sqrt{2}}{9}$

$= \frac{\cancel{2}^{1}ab}{{}_{9}\cancel{18}} = \frac{4\sqrt{2}}{9}$

$\Rightarrow \frac{ab}{9} = \frac{4\sqrt{2}}{9}$

$\Rightarrow ab = 4\sqrt{2}$ -④

now solving ① and ③ simultaneously (eliminating $a^2$)

$a^2+b^2 = 18$

$-\ a^2-b^2 = 14$

$2b^2 = 4$

$\div 2 \quad b^2 = 2 \quad \div 2$

$\Rightarrow b = \pm\sqrt{2}$

sub this into (1)

$a^2 + (\pm\sqrt{2})^2 = 18$

$a^2 + 2 = 18$

$a^2 = 16$

$\Rightarrow a = \pm\sqrt{16}$

$= \pm 4$

this suggests that there are 4 possible complex number combinations, but checking with ④ to see if all work with it as well

$ab = 4\sqrt{2}$

$\therefore$ either $a = 4, b = \sqrt{2}$

OR $a = -4, b = -\sqrt{2}$

$\boxed{\therefore z = 4+\sqrt{2}i \text{ or } -4-\sqrt{2}i}$

METHOD 2: using MOD-ARG form

know that $zz^* = 18$

now trying to evaluate

$\frac{z}{z^*} \times \frac{z}{z}$ (rationalising denominator)

$= \frac{z^2}{zz^*} = \frac{z^2}{18}$

$\therefore \frac{z^2}{18} = \frac{7}{9} + \frac{4\sqrt{2}}{9}i$ $\times 18$ $\times 18$

$\Rightarrow z^2 = 14 + 8\sqrt{2}i$

**Question 7 continued**

turning $z^2$ into mod-arg form

$r = |z| = \sqrt{(14)^2 + (8\sqrt{2})^2}$

$= \sqrt{196 + 128}$

$= 18$

$\therefore z^2 = 18(\cos\alpha + i\sin\alpha)$

where $\tan\alpha = \frac{b}{a} = \frac{8\sqrt{2}}{14} = \frac{4\sqrt{2}}{7}$

hence square rooting to get $z$

(means have to halve angle in bracket)

~~$z = \pm\sqrt{18}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$~~

key here is to find values of $\cos\frac{\alpha}{2}$ and $\sin\frac{\alpha}{2}$

... starting with $\cos\frac{\alpha}{2}$, recognise this as part of the cos double angle result

$\cos\alpha = 2\cos^2\tfrac{1}{2}\alpha - 1$

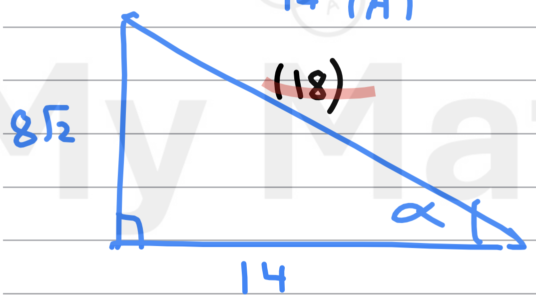
... rearrange to get $\cos^2\tfrac{1}{2}\alpha$ :

$\cos^2\tfrac{1}{2}\alpha = \frac{\cos\alpha + 1}{2}$

$\therefore \cos\tfrac{1}{2}\alpha = \pm\sqrt{\tfrac{1}{2}(1 + \cos\alpha)}$

and we can find $\cos\alpha$ using the $\tan\alpha$ above

$\tan\alpha = \frac{8\sqrt{2}\ (O)}{14\ (A)}$




$\therefore \cos\alpha = \frac{A}{H} = \frac{14}{18} = \frac{7}{9}$

$\therefore H = \sqrt{(8\sqrt{2})^2 + (14)^2}$

$= 18$

subbing this result into expression for $\cos\tfrac{1}{2}\alpha$

$\cos\frac{\alpha}{2} = \pm\sqrt{1 + 7/9}$

$= \pm\sqrt{8/9}$

$= \frac{2\sqrt{2}}{3}$

now we need to find $\sin^2\frac{\alpha}{2}$, which can be found using the Pythagorean identity: $\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}=1$

$$\Rightarrow \sin^2\frac{\alpha}{2}+\left(\frac{2\sqrt{2}}{3}\right)^2=1$$
$$\Rightarrow \sin^2\frac{\alpha}{2}=1-8/9$$
$$\Rightarrow \sin^2\frac{\alpha}{2}=1/9$$
$$\Rightarrow \sin\frac{\alpha}{2}=\pm 1/3$$

Now we can sub into the prev. expression for $z$

$$z=\pm\sqrt{18}\left(\frac{2\sqrt{2}}{3}+\frac{1}{3}i\right)$$

$\times\sqrt{18}$ $\times\sqrt{18}$

$$\boxed{\Rightarrow z=\pm(4+\sqrt{2}i)}$$

METHOD 3: using EXPONENTIAL FORM

know that $zz^*=18$

$a^2+b^2=18$ — ①

now convert

$$a+bi=re^{i\theta}=r(\cos\theta+i\sin\theta)$$
$$a-bi=re^{-i\theta}=r(\cos\theta-i\sin\theta)$$

using $\frac{z}{z^*}=\frac{\cancel{r}e^{i\theta}}{\cancel{r}e^{-i\theta}}=e^{i2\theta}$

$$=\cos(2\theta)+i\sin(2\theta)$$

∴ equating to RHS

$$\cos 2\theta+i\sin 2\theta=\frac{7}{9}+\frac{4\sqrt{2}}{9}i$$

equating real components

$Re\left(\frac{z}{z^*}\right)$: $\cos 2\theta=7/9$

using cos double angle formula to get $\cos\theta$ and $\sin\theta$

$$2\cos^2\theta-1=7/9$$
$$2\cos^2\theta=16/9$$
$\div 2$ $\div 2$
$$\cos^2\theta=8/9$$
$$\cos^2\theta=\pm\sqrt{8/9}$$

∴ getting $\sin\theta$ by using Pythagorean identity

$$\cos^2\theta+\sin^2\theta=1$$
$$\left(\pm\sqrt{8/9}\right)^2+\sin^2\theta=1$$
$$\sin^2\theta=1-8/9$$
$$=1/9$$
$$\sin\theta=\pm\sqrt{1/9}$$
$$=\pm 1/3$$

**Question 7 continued**

now sub into prev. equation for $z$

$z = r\left(\frac{2\sqrt{2}}{3} + \frac{1}{3}i\right)$

where $r = |z| = \sqrt{a^2+b^2}$

$= \sqrt{18}$

$= \pm\sqrt{18}\left(\frac{2\sqrt{2}}{3} + \frac{1}{3}i\right)$

$\boxed{= \pm(4+\sqrt{2}i)}$

**(Total for Question 7 is 7 marks)**

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**8.** (a) Given

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad n \in \mathbb{N}$$

show that

$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

**(5)**

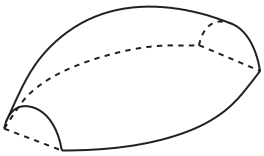


**Figure 1**

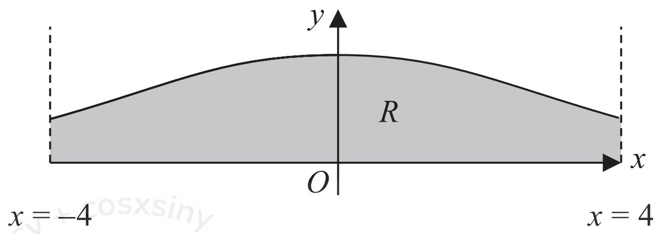



**Figure 2**

Figure 1 shows a solid paperweight with a flat base.

Figure 2 shows the curve with equation

$$y = H\cos^3\left(\frac{x}{4}\right) \qquad -4 \leqslant x \leqslant 4$$

where $H$ is a positive constant and $x$ is in radians.

The region $R$, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = -4$, the line with equation $x = 4$ and the $x$-axis.

The paperweight is modelled by the solid of revolution formed when $R$ is rotated **180°** about the $x$-axis.

Given that the maximum height of the paperweight is 2 cm,

(b) write down the value of $H$.

**(1)**

(c) Using algebraic integration and the result in part (a), determine, in $\text{cm}^3$, the volume of the paperweight, according to the model. Give your answer to 2 decimal places.

[*Solutions based entirely on calculator technology are not acceptable.*]

**(5)**

(d) State a limitation of the model.

**(1)**

(a) the question is asking to express the trig power of $32\cos^6\theta$ as multiple angle functions ($\cos n\theta$)

so first re-write $\cos^6\theta$ straight away using given identity that

$z^n + z^{-n} = 2\cos\theta$

**Question 8 continued**

i.e $\left(z+\frac{1}{z}\right)^6 = (2\cos\theta)^6$

$= 64\cos^6\theta = R.H.S$

now want to expand the L.H.S (Binomial expansion)

$(z+z^{-1})^6 = \binom{6}{0}z^6(1/z)^0 + \binom{6}{1}z^5(1/z)^1 + \binom{6}{2}z^4(1/z)^2 + \binom{6}{3}z^3(1/z)^3 +$

$\binom{6}{4}z^2(1/z)^4 + \binom{6}{5}z(1/z)^5 + \binom{6}{6}z^0(1/z)^6$

and simplifying

$z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$

collecting like terms

$(z^6 + z^{-6}) + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20$

...using prev. identity that $z^n + z^{-n} = 2\cos n\theta$

$(2\cos 6\theta) + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$

$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

now equating to R.H.S

$2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20 = 64\cos^6\theta$

but in question have expression for $32\cos^6\theta$

$\therefore$ need to divide by 2

$\div 2$ $\div 2$

$$\boxed{32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 \quad \text{as required}}$$

(b) notice this is a 'modelling' volumes of revolution question

↳ using the given information that $y_{max} = 2$ and the curve

$y = H\cos^3\left(\frac{x}{4}\right)$

$y$ is max. where $\cos^3\left(\frac{x}{4}\right)$ is largest

(where $\cos\theta = 1$)

$\therefore \cos^3\left(\frac{x}{4}\right) = 1$

$y_{max} = H(1)$

$= 2(1)$

$$\boxed{\therefore H = 2}$$

(c) remembering the formula for volumes of revolution 360° about the x-axis

$$V = \pi \int y^2 \, dx$$

but here we have volumes of revolution 180° about x-axis

$\therefore$ need half of above

$$V = \frac{\pi}{2} \int y^2 \, dx$$

here $y = 2\cos^3\left(\frac{x}{4}\right)$

$y^2 = 4\cos^6\left(\frac{x}{4}\right)$

subbing into above equ. and taking coefficients out in front

$$V = \frac{4\pi}{2} \int \cos^6\left(\frac{x}{4}\right) dx$$

$$\Rightarrow V = 2\pi \int \cos^6\left(\frac{x}{4}\right) dx$$

rewriting $\cos^6\left(\frac{x}{4}\right)$ as part (a) $\rightarrow \div 32$ and $\theta = x/4$

$$= \frac{2\pi}{32} \int \left(\cos\left(6 \times \frac{x}{4}\right) + 6\cos\left(4 \times \frac{x}{4}\right) + 15\cos\left(2 \times \frac{x}{4}\right) + 10\right) dx$$

$$= \frac{\pi}{16} \int \left(\cos\left(\tfrac{3}{2}x\right) + 6\cos(x) + 15\cos\left(\frac{x}{2}\right) + 10\right) dx$$

putting in the limits - but here easier to find integral $0 \leq x \leq 4$ and $\times 2$ (due to symmetry of curve)

$$\frac{\pi}{16} \int_0^4 \left(\cos\left(\tfrac{3}{2}x\right) + 6\cos(x) + 15\cos\left(\frac{x}{2}\right) + 10\right) dx$$

$$\frac{\pi}{16} \left[ \tfrac{2}{3}\sin\left(\tfrac{3}{2}x\right) + 6\sin(x) + 30\sin\left(\frac{x}{2}\right) + 10x \right]_0^4$$

using $\int f'(ax+b)\,dx = \frac{1}{a} f(ax+b) + c$

$$\frac{\pi}{16} \left\{ \left[ \tfrac{2}{3}\sin(6) + 6\sin(4) + 30\sin(2) + 10(4) \right] - [0] \right\}$$

...evaluating on calc:

$= 12.28 \times 2$

$\boxed{= 24.56\text{ cm}^3}$

(d) possible LIMITATIONS

- equation of the line may not be suitable
- measurements may not be accurate
- paperweight may not be smooth

**Question 8 continued**

**(Total for Question 8 is 12 marks)**

**9.** (i) (a) Explain why $\displaystyle\int_0^{\infty} \cosh x \, dx$ is an improper integral.

**(1)**

(b) Show that $\displaystyle\int_0^{\infty} \cosh x \, dx$ is divergent.

**(3)**

(ii) $$4\sinh x = p\cosh x \qquad \text{where } p \text{ is a real constant}$$

Given that this equation has real solutions, determine the range of possible values for $p$

**(2)**

(i)(a) considering the 2 possible cases where integral is 'improper'

CASE 1 : when there is a '∞' in one of the limits

CASE 2 : when integrand is discontinuous across the given interval

here, see that there is a '∞' in the limit, which is why it's improper

(b) even when the integral is improper, if we get it to converge to a finite area, then we can find the integral

but if the function diverges, its integral cannot be integrated

here we need to show that integral diverges when $t \to \infty$

evaluating the integral

$$\int_0^{\infty} \cosh x \, dx = \lim_{t\to\infty} \int_0^t \cosh x \, dx$$

WAY 1 : using hyperbolic functions integration

$$= \lim_{t\to\infty} [\sinh x]_0^t$$

$$= \lim_{t\to\infty} \{\sinh t - \sinh 0\}$$

$$= \lim_{t\to\infty} \{\sinh t\}$$

as $t \to \infty$, $\sinh t \to \infty$

$$\boxed{\therefore \int_0^{\infty} \cosh x \, dx \text{ is divergent}}$$

WAY 2 : using EXPONENTIAL DEFINITION of $\cosh x$

$$\lim_{t\to\infty} \int_0^t \frac{1}{2}(e^x + e^{-x}) \, dx$$

**Question 9 continued**

$$\lim_{t\to\infty} \frac{1}{2}\int_0^t (e^x + e^{-x})\,dx$$

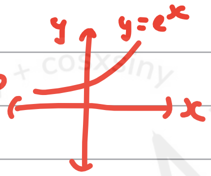
$$\lim_{t\to\infty} \frac{1}{2}\left[e^x - e^{-x}\right]_0^t$$

$$\lim_{t\to\infty} \frac{1}{2}\left\{e^t - e^{-t} - (e^0 - e^{-0})\right\}$$

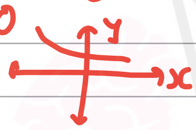
$$\lim_{t\to\infty} \frac{1}{2}\left\{e^t - e^{-t} - (1-1)\right\}$$

$$\lim_{t\to\infty} \frac{1}{2}e^t - \frac{1}{2}e^{-t}$$

as $t\to\infty$, $e^t\to\infty$




as $t\to\infty$, $e^{-t}\to 0$




hence

$$\boxed{\int_0^\infty \cosh x\,dx \text{ is DIVERGENT}}$$

(ii) $4\sinh x = p\cosh x$

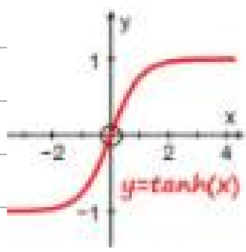
start by $\div \cosh x$

$4\tanh x = p$

$\tanh x = \frac{p}{4}$

need REAL solutions

$\therefore$ considering $y = \tanh x$ function (below)




for real solutions need

$-1 < \tanh x < 1$

$-1 < p/4 < 1$

$$\boxed{-4 < p < 4}$$

**(Total for Question 9 is 6 marks)**

**10.**

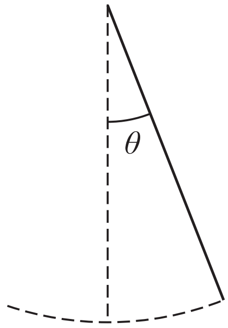



**Figure 3**

The motion of a pendulum, shown in Figure 3, is modelled by the differential equation

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

where $\theta$ is the angle, in radians, that the pendulum makes with the downward vertical, $t$ seconds after it begins to move.

(a) (i) Show that a particular solution of the differential equation is

$$\theta = \frac{1}{12}t\sin 3t$$

**(4)**

(ii) Hence, find the general solution of the differential equation.

**(4)**

Initially, the pendulum

- makes an angle of $\frac{\pi}{3}$ radians with the downward vertical
- is at rest

Given that, 10 seconds after it begins to move, the pendulum makes an angle of $\alpha$ radians with the downward vertical,

(b) determine, according to the model, the value of $\alpha$ to 3 significant figures.

**(4)**

Given that the true value of $\alpha$ is 0.62

(c) evaluate the model.

**(1)**

The differential equation

$$\frac{d^2\theta}{dt^2} + 9\theta = \frac{1}{2}\cos 3t$$

models the motion of the pendulum as moving with forced harmonic motion.

(d) Refine the differential equation so that the motion of the pendulum is simple harmonic motion.

**(1)**

**Question 10 continued**

(a)(i) can show that this is the particular integral (need to find this when dealing with non-homogenous 2.O.D.Es) in 2 ways

WAY 1 : differentiating given the P.I to get equation

$$\theta = \frac{1}{12} t \sin 3t$$

using product rule

$$\frac{d\theta}{dt} = \frac{1}{12} t (3\cos 3t) + \frac{1}{12} \sin 3t$$
$$= \frac{1}{4} t \cos 3t + \frac{1}{12} \sin 3t$$

using product rule

$$\frac{d^2\theta}{dt^2} = \frac{1}{4} t (-3\sin 3t) + \frac{1}{4}\cos 3t + \frac{3}{12}\cos 3t$$
$$= -\tfrac{3}{4} t \sin 3t + \frac{1}{2}\cos 3t$$

$\therefore$ subbing into equation to see if get R.H.S

$$-\tfrac{3}{4} \cancel{t\sin 3t} + \frac{1}{2}\cos 3t + \tfrac{3}{4} \cancel{t \sin 3t}$$
$$= \frac{1}{2}\cos 3t = \text{R.H.S}$$

$\boxed{\therefore \text{ this is the particular solution the 2.O.D.E}}$

WAY 2 : finding the P.I

Looking at the given particular solution, we can see that 't' is present in order to prevent a multiple of the C.F being present in the P.I

| Form of f(x) | Form of particular integral |
|---|---|
| $k$ | $\lambda$ |
| $ax + b$ | $\lambda + \mu x$ |
| $ax^2 + bx + c$ | $\lambda + \mu x + \nu x^2$ |
| $ke^{px}$ | $\lambda e^{px}$ |
| $m \cos \omega x$ | $\lambda \cos \omega x + \mu \sin \omega x$ |
| $m \sin \omega x$ | $\lambda \cos \omega x + \mu \sin \omega x$ |
| $m \cos \omega x + n \sin \omega x$ | $\lambda \cos \omega x + \mu \sin \omega x$ |

**WARNING!**
The particular integral must not contain any term in the complementary function. If it does, you'll need to add an x and possibly even an $x^2$ in front of your usual PI form

hence let $\theta = \lambda t \sin 3t$

product rule

$$\frac{d\theta}{dt} = \lambda \sin 3t + 3\lambda t \cos 3t$$
$$\frac{d^2\theta}{dt^2} = 3\lambda \cos 3t + 3\lambda \cos 3t$$
$$= -9\lambda t \sin 3t$$
$$= 6\lambda \cos 3t - 9\lambda t \sin 3t$$

Subbing into the given 2.O.D.E

$$6\lambda\cos 3t - 9\cancel{\lambda t \sin 3t} + 9\cancel{\lambda t \sin 3t} = \frac{1}{2}\cos 3t$$
$$\Rightarrow 6\lambda \cos 3t = \frac{1}{2}\cos 3t$$

comparing coefficients

Question 10 continued

$$6\alpha = \tfrac{1}{2}$$
$\div 6$ $\quad \alpha = \tfrac{1}{12}$ $\quad \div 6$

$\boxed{\therefore P.I = \tfrac{1}{12} t\sin 3t \text{ as required}}$

(a)(ii) remember G.S = C.F + P.I

we already know the P.I, so let's work out C.F

$A.E: m^2 + 9 = 0$

$\Rightarrow m = \pm\sqrt{-9}$

$\Rightarrow m = \pm 3i$

using the 'imaginary solutions' form i.e when $\pm wi$,

G.S : $y = A\cos wt + B\sin wt$

$C.F = A\cos 3t + B\sin 3t$

$\boxed{\therefore G.S = A\cos 3t + B\sin 3t + \tfrac{1}{12} t\sin 3t}$

(b) G.S represents a whole family of solutions where different 'A' and 'B' values represent different particular solutions

here we are looking at the following particular conditions:

when $t=0$, $\theta = \pi/3$

when $t=0$, $\frac{d\theta}{dt} = 0$

subbing these into G.S

$\pi/3 = A\cos(0) + B\sin(0) + \frac{1}{2}(0)\sin(0)$

$\Rightarrow A = \pi/3$

product rule

$\frac{d\theta}{dt} = -3A\sin 3t + 3B\cos 3t + \frac{1}{12}\sin 3t + \frac{1}{4}t\cos 3t$

$0 = -3A(\sin(0)) + 3B(0) + \frac{1}{12}(0) + \frac{1}{4}(0)(\cos(0))$

$\Rightarrow 0 = 3B$

$\Rightarrow B = 0$

$\therefore$ know that P.S: $\theta = \pi/3 \cos 3t + \frac{1}{12} t\sin 3t$

now need $\theta$ when $t = 10$

$= \pi/3 \cos 3(10) + \frac{1}{12}(10)(\sin(30))$

$= \pi/3 (\sqrt{3}/2) + \frac{5}{6}(1/2)$

$\Rightarrow \theta = -0.661827..$

$\boxed{\therefore \alpha = -0.662}$ (3s.f),,





DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

**Question 10 continued**

(c) 0.662 is close to 0.62 $\boxed{\therefore \text{ good model at } t=10}$

( % error is $\frac{0.66182.. - 0.62}{0.62} \times 100 = 6.745\% < 10\%$

$\boxed{\therefore \text{ suitable model}}$ ) //

(d) knowing the S.H.M equation as $\ddot{x} + w^2 x = 0$

need to adjust our non-homogenous 2.O.D.E to

$$\boxed{\frac{d^2\theta}{d\theta^2} + 9\theta = 0}$$

**Question 10 continued**

**(Total for Question 10 is 14 marks)**

**TOTAL FOR PAPER IS 75 MARKS**